

# Cramer's rules for Hermitian systems of coquaternionic equations.

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## Abstract

In this paper properties of the determinant of a Hermitian matrix are investigated, and determinantal representations of the inverse of a Hermitian coquaternionic matrix are given. By their using, Cramer's rules for left and right systems of linear equations with Hermitian coquaternionic matrices of coefficients are obtained. Cramer's rule for a two-sided coquaternionic matrix equation  $\mathbf{AXB} = \mathbf{D}$  (with Hermitian  $\mathbf{A}, \mathbf{B}$ ) is given as well.

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## 1 Introduction

A quaternion algebra  $\mathbf{H}(a, b)$  over a field  $\mathbf{F}$  (denoted by  $(\frac{a, b}{\mathbf{F}})$ ) are a central simple algebra over  $\mathbf{F}$ , and a four-dimensional vector space over  $\mathbf{F}$  with basis  $\{1, i, j, k\}$  and the following multiplication rules:

$$i^2 = a, \quad j^2 = b, \quad ij = k, \quad ji = -k,$$

where  $\{a, b\} \subset \mathbf{F}$ . To every quaternion algebra  $\mathbf{H}(a, b)$ , one can associate a quadratic form  $\mathbf{n}$  (called the norm form) on  $\mathbf{H}$  such that  $\mathbf{n}(xy) = \mathbf{n}(x)\mathbf{n}(y)$ ,

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for all  $x$  and  $y$  in  $\mathbf{H}$ . A linear mapping  $x \rightarrow \bar{x} = \mathbf{t}(x) - x$  is also defined on  $\mathbf{H}$ . It is an involution with properties

$$\bar{\bar{x}} = x, \overline{x + y} = \bar{x} + \bar{y}, \overline{x \cdot y} = \bar{y} \cdot \bar{x}.$$

An element  $\bar{x}$  is called the conjugate of  $x \in \mathbf{H}$ .  $\mathbf{t}(x)$  and  $\mathbf{n}(x)$  are called the trace and the norm of  $x$  respectively.  $\{\mathbf{n}(x), \mathbf{t}(x)\} \subset \mathbf{F}$  for all  $x$  in  $\mathbf{H}$  and possess the following conditions,

$$\mathbf{n}(\bar{x}) = \mathbf{n}(x), \mathbf{t}(\bar{x}) = \mathbf{t}(x), \mathbf{t}(q \cdot p) = \mathbf{t}(p \cdot q).$$

Depending on the choice of  $\mathbf{F}$ ,  $a$  and  $b$  we have only two possibilities ([1]):

1.  $(\frac{a,b}{\mathbf{F}})$  is a division algebra. The most famous example of a non-split quaternion algebra is Hamilton's quaternions  $\mathbb{H} = (\frac{-1,-1}{\mathbb{R}})$ .
2.  $(\frac{a,b}{\mathbf{F}})$  is isomorphic to the algebra of all  $2 \times 2$  matrices with entries from  $\mathbf{F}$ . In this case we say that the  $\mathbf{F}$ -algebra is split.

In contrast to a quaternion division algebra, a split quaternion algebra contains zero-divisors, nilpotent elements and nontrivial idempotents.

One of the most famous split quaternion algebras is the split quaternions of James Cockle ([2])  $\mathbf{H}_S(\frac{-1,1}{\mathbb{R}})$ , which can be represented as

$$\mathbf{H}_S = \{q = q_0 + q_1i + q_2j + q_3k : \{q_0, q_1, q_2, q_3\} \in \mathbb{R}\}.$$

$\mathbf{H}_S$  is an associative, non-commutative, non-division ring with four basis elements  $\{1, i, j, k\}$  satisfying the equalities

$$\begin{aligned} i^2 &= -1, j^2 = k^2 = 1, \\ ij &= -ji = k, jk = -kj = -i, ik = -ki = -j. \end{aligned}$$

The split quaternions of James Cockle are also named coquaternions. In this paper we shall consider coquaternions and denote their  $\mathbf{H}$  to simplify.

Coquaternions is a recently developing topic. There are some studies related to geometric applications of split quaternions such as ([3]-[5]). Particularly, the geometric and physical applications of coquaternions require solving coquaternionic equations and their systems. Therefore, there are many studies on coquaternionic equations. We mention only some recent papers. The method of rearrangements has been used to solve linear quaternionic equations involving  $axb$  in [6], new method of solving general linear

coquaternionic equations with the terms of the form  $axb$  has been obtained in [7]. The properties of coquaternion matrices has been discussed in [8]. Particularly, in [8] the authors have defined the complex adjoint matrix of coquaternion matrices and given the definition of q-determinant of coquaternion matrices that is an usual determinant of the complex adjoint matrix.

Recently, in [9] the concept of immanant (consequently, determinant and permanent) has been extended to a split quaternion algebra using methods of the theory of the row and column determinants. The theory of the row and column determinants was introduced in [10, 11] for matrices over the quaternion non-split algebra. This theory over the quaternion skew field is being actively developed as by the author (see, for ex.[12]-[15]), and others (see, for ex. [16]-[18]).

In this paper properties of the determinant of a Hermitian matrix over  $\mathbf{H}$  will be investigated, and determinantal representations will be given for the inverse of a Hermitian coquaternion matrix. Firstly in Section 2, we shall give some properties of coquaternions, coquaternion matrices, and noncommutative determinants in Subsection 2.1, and some basic concepts and results from the theory of the row-column determinants of coquaternion matrices in Subsection 2.2. We shall consider the lemma about expanding row and column determinants by cofactors along corresponding rows and columns in this subsection as well. In Section 3, properties of the determinant of a Hermitian coquaternion matrix will be investigated by using row-column determinants. In Section 4, determinantal representations for inverses of Hermitian coquaternion matrix will be given and Cramer' rules for left and right systems of linear equations will be obtained. In Section 5, we shall get Cramer's rule for two-sided coquaternionic matrix equations  $\mathbf{AXB} = \mathbf{D}$ , where  $\mathbf{A}$ ,  $\mathbf{B}$  are Hermitian. The main results will be illustrated by examples.

## 2 Preliminaries

### 2.1 Coquaternions, coquaternion matrices and noncommutative determinants

For any coquaternion  $q = q_0 + q_1i + q_2j + q_3k \in \mathbf{H}$ , by  $\text{Re } q := q_0$  and  $\text{Im } q := q_1i + q_2j + q_3k$ , we define the real and imagine parts of  $q$ , respectively. The conjugate of a coquaternion  $q$  is  $\bar{q} = q_0 - q_1i - q_2j - q_3k$ , then the trace

$\mathbf{t}(q) = 2\operatorname{Re} q = 2q_0$  and the norm form  $\mathbf{n}(q) = q\bar{q} = q_0^2 + q_1^2 - q_2^2 - q_3^2$ . The norm form of an coquaternion  $q$  usually denote by  $I_q := \mathbf{n}(q)$ . The norm of a coquaternion by  $\|q\| = \sqrt{|I_q|}$  are considered as well. If  $\|q\| = 1$ , then  $q$  is called unit coquaternion. Notice that  $p = \frac{q}{\|q\|}$  is a unit coquaternion for  $q \in \mathbf{H}$  with  $\|q\| \neq 0$  and 1,  $i, j$  and  $k$  are the basis units. The inverse of the coquaternion  $q$  is  $q^{-1} = \frac{\bar{q}}{I_q}$ , where  $I_q \neq 0$ . We indicate by  $\mathcal{U}(\mathbf{H})$  the set of all invertible elements of  $\mathbf{H}$  and  $\mathcal{D}(\mathbf{H})$  the set of all zero-divisors of  $\mathbf{H}$ .

Denote by  $\mathbf{H}^{n \times m}$  a set of  $n \times m$  matrices with entries in  $\mathbf{H}$  and by  $M(n, \mathbf{H})$  a ring of  $n \times n$  coquaternionic matrices. This is a ring with a unit which is the usual identity matrix  $\mathbf{I}_n$ . By usual way, we define the transpose  $\mathbf{A}^T = (a_{ji}) \in \mathbf{H}^{n \times m}$ , the conjugate  $\bar{\mathbf{A}} = (\bar{a}_{ij}) \in \mathbf{H}^{m \times n}$ , the Hermitian adjoint matrix (the conjugate transpose)  $\mathbf{A}^* = (\bar{a}_{ji}) \in \mathbf{H}^{n \times m}$  of  $\mathbf{A} = (a_{ij}) \in \mathbf{H}^{m \times n}$ , and the inverse  $\mathbf{A}^{-1}$  of  $\mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n}$ . For more properties of split quaternions the reader is referred to [19]-[21].

Definition of determinant of matrices with noncommutative entries (that are also defined as noncommutative determinants) is more associated with matrices over the skew field of Hamilton's quaternions  $\mathbb{H}$ . There are even three approaches in its defining. The first approach to defining the determinant of a matrix in  $M(n, \mathbb{H})$  is as follows [22, 23].

**Definition 2.1** *Let a functional  $d : M(n, \mathbb{H}) \rightarrow \mathbb{H}$  satisfy the following axioms.*

**Axiom 1**  $d(\mathbf{A}) = 0$  if and only if the matrix  $\mathbf{A}$  is non invertible.

**Axiom 2**  $d(\mathbf{A} \cdot \mathbf{B}) = d(\mathbf{A}) \cdot d(\mathbf{B})$  for  $\forall \mathbf{B} \in M(n, \mathbb{H})$ .

**Axiom 3** *If the matrix  $\mathbf{A}'$  is obtained from  $\mathbf{A}$  by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then  $d(\mathbf{A}') = d(\mathbf{A})$ .*

*Then the functional  $d$  is called the determinant of  $\mathbf{A} \in M(n, \mathbb{H})$ .*

But in [22], it is proved that if a determinant functional satisfies Axioms 1, 2, 3, then its value is real. The famous examples of such determinant are the determinants of Study and Diedonné.

In another way of looking a noncommutative determinant is defined as a rational function from entries. In particular, in the theory of the Gelfand-Retah quasideterminants [24, 25], an arbitrary  $n \times n$  matrix over a skew field

has been associated with an  $n \times n$  matrix whose entries are quasideterminants. The quasideterminant is not an analog of the usual determinant but rather of a ratio of the determinant of an  $n \times n$ -matrix to the determinant of an  $(n-1) \times (n-1)$ -submatrix.

At last, at the third approach a noncommutative determinant is defined, by analogy to the usual determinant, as the alternating sum of  $n!$  products of entries of a matrix but by specifying a certain ordering of coefficients in each term. Moore [26] was the first who achieved the fulfillment of the main Axiom 1 by such definition of a noncommutative determinant. But it has been done not for all square matrices over a skew field but only Hermitian matrices. Later, Dyson [27] described the theory in more modern terms. But until recently, the Moore determinant has not been extended to arbitrary square matrices over  $\mathbb{H}$ . The full and natural extension of the definition of Moore's determinant to arbitrary square matrices over  $\mathbb{H}$  has been reached in the theory of column-row determinants.

Recently in [8] the  $q$ -determinant of coquaternionic matrices has been introduced by the follows. Let  $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 j \in \mathbf{H}^{n \times n}$ , where  $\mathbf{A}_1$  and  $\mathbf{A}_2$  are complex matrices. Then the complex adjoint matrix  $\chi_A \in \mathbb{C}^{2n \times 2n}$  is defined as

$$\chi_A := \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \overline{\mathbf{A}_2} & \overline{\mathbf{A}_1} \end{pmatrix}$$

and the  $q$ -determinant of  $\mathbf{A}$  is defined as the usual determinant of  $\chi_A$ , that is  $|\mathbf{A}|_q = |\chi_A|$ . It has been shown that properties of the  $q$ -determinant is close to the usual determinant, especially, it satisfies Axioms 1, 2. Since the  $q$ -determinant of  $\mathbf{A} \in \mathbf{H}^{n \times n}$  takes a value not in  $\mathbf{H}$  but in  $\mathbb{C}$  and the  $q$ -determinant can not be expanded by cofactors along an arbitrary row or column, then determinantal representations of the inverse  $\mathbf{A}^{-1}$  by the  $q$ -determinant could not be obtained.

## 2.2 Definitions and basic properties of the column and row determinants

For  $\mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n}$  we define  $n$  row determinants as follows.

**Definition 2.2** [9] *The  $i$ -th row determinant of  $\mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n}$  is defined as*

$$\text{rdet}_i \mathbf{A} = \sum_{\sigma \in S_n} (-1)^{n-r} a_{i i_{k_1}} a_{i_{k_1} i_{k_1+1}} \dots a_{i_{k_1+l_1} i} \dots a_{i_{k_r} i_{k_r+1}} \dots a_{i_{k_r+l_r} i_{k_r}},$$

where left-ordered cycle notation of the permutation  $\sigma$  is written as follows

$$\sigma = (i \ i_{k_1} i_{k_1+1} \dots i_{k_1+l_1}) (i_{k_2} i_{k_2+1} \dots i_{k_2+l_2}) \dots (i_{k_r} i_{k_r+1} \dots i_{k_r+l_r}).$$

Here the index  $i$  starts the first cycle from the left and other cycles satisfy the conditions,  $i_{k_2} < i_{k_3} < \dots < i_{k_r}$ ,  $i_{k_t} < i_{k_t+s}$ , for all  $t = \overline{2, r}$  and  $s = \overline{1, l_t}$ , (since  $\text{sign}(\sigma) = (-1)^{n-r}$ ).

For  $\mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n}$  we define  $n$  column determinant as well.

**Definition 2.3** [9] The  $j$ -th column determinant of  $\mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n}$  is defined as

$$\text{rdet}_j \mathbf{A} = \sum_{\tau \in S_n} (-1)^{n-r} a_{j_{k_r} j_{k_r+l_r}} \dots a_{j_{k_r+1} j_{k_r}} \dots a_{j_{k_1+l_1} j_{k_1}} \dots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j},$$

where right-ordered cycle notation of the permutation  $\tau \in S_n$  is written as follows

$$\tau = (j_{k_r+l_r} \dots j_{k_r+1} j_{k_r}) \dots (j_{k_2+l_2} \dots j_{k_2+1} j_{k_2}) (j_{k_1+l_1} \dots j_{k_1+1} j_{k_1} j).$$

Here the first cycle from the right begins with the index  $j$  and other cycles satisfy the following conditions,  $j_{k_2} < j_{k_3} < \dots < j_{k_r}$ ,  $j_{k_t} < j_{k_t+s}$ , for all  $t = \overline{2, r}$  and  $s = \overline{1, l_t}$

In [9] the basic properties of the column and row immanants of a square matrix over  $\mathbf{H}$  has been consider. These properties can be evidently extend to column-row determinants.

**Proposition 2.4** (The first theorem about zero of an row-column determinant) If one of the rows (columns) of  $\mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n}$  consists of zeros only, then  $\text{rdet}_i \mathbf{A} = 0$  and  $\text{cdet}_i \mathbf{A} = 0$  for all  $i = \overline{1, n}$ .

Denote by  $\mathbf{H}\mathbf{a}$  and  $\mathbf{a}\mathbf{H}$  left and right principal ideals of  $\mathbf{H}$ , respectively.

**Proposition 2.5** (The second theorem about zero of an row determinant) Let  $\mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n}$  and  $a_{ki} \in \mathbf{H}a_i$  and  $a_{ij} \in \overline{a_i}\mathbf{H}$ , where  $n(a_i) = 0$  for  $k, j = \overline{1, n}$  and for all  $i \neq k$ . Let  $a_{11} \in \mathbf{H}a_1$  and  $a_{22} \in \overline{a_1}\mathbf{H}$  if  $k = 1$ , and  $a_{kk} \in \mathbf{H}a_k$  and  $a_{11} \in \overline{a_k}\mathbf{H}$  if  $k = i > 1$ , where  $n(a_k) = 0$ . Then  $\text{rdet}_k \mathbf{A} = 0$ .

**Proposition 2.6** (The second theorem about zero of a column determinant) Let  $\mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n}$  and  $a_{ik} \in a_i \mathbf{H}$  and  $a_{ji} \in \mathbf{H} \overline{a_i}$ , where  $n(a_i) = 0$  for  $k, j = \overline{1, n}$  and for all  $i \neq k$ . Let  $a_{11} \in a_1 \mathbf{H}$  and  $a_{22} \in \mathbf{H} \overline{a_1}$  if  $k = 1$ , and  $a_{kk} \in a_k \mathbf{H}$  and  $a_{11} \in \mathbf{H} \overline{a_k}$  if  $k = i > 1$ , where  $n(a_k) = 0$ . Then  $\text{cdet}_k \mathbf{A} = 0$ .

**Proposition 2.7** If the  $i$ -th row of  $\mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n}$  is left-multiplied by  $b \in \mathbf{H}$ , then  $\text{rdet}_i \mathbf{A}_i. (b \cdot \mathbf{a}_i.) = b \cdot \text{rdet}_i \mathbf{A}$  for all  $i = \overline{1, n}$ .

**Proposition 2.8** If the  $j$ -th column of  $\mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n}$  is right-multiplied by  $b \in \mathbf{H}$ , then  $\text{cdet}_j \mathbf{A}_{.j} (\mathbf{a}_{.j} \cdot b) = \text{cdet}_j \mathbf{A} \cdot b$  for all  $j = \overline{1, n}$ .

**Proposition 2.9** If for  $\mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n}$  there exists  $t \in \{1, \dots, n\}$  such that  $a_{tj} = b_j + c_j$  for all  $j = \overline{1, n}$ , then for all  $i = \overline{1, n}$

$$\begin{aligned} \text{rdet}_i \mathbf{A} &= \text{rdet}_i \mathbf{A}_{t.} (\mathbf{b}) + \text{rdet}_i \mathbf{A}_{t.} (\mathbf{c}), \\ \text{cdet}_i \mathbf{A} &= \text{cdet}_i \mathbf{A}_{t.} (\mathbf{b}) + \text{cdet}_i \mathbf{A}_{t.} (\mathbf{c}), \end{aligned}$$

where  $\mathbf{b} = (b_1, \dots, b_n) \in \mathbf{H}^{1 \times n}$ ,  $\mathbf{c} = (c_1, \dots, c_n) \in \mathbf{H}^{1 \times n}$  are arbitrary row-vectors.

**Proposition 2.10** If for  $\mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n}$  there exists  $t \in \{1, \dots, n\}$  such that  $a_{it} = b_i + c_i$  for all  $i = \overline{1, n}$ , then for all  $j = \overline{1, n}$

$$\begin{aligned} \text{rdet}_j \mathbf{A} &= \text{rdet}_j \mathbf{A}_{.t} (\mathbf{b}) + \text{rdet}_j \mathbf{A}_{.t} (\mathbf{c}), \\ \text{cdet}_j \mathbf{A} &= \text{cdet}_j \mathbf{A}_{.t} (\mathbf{b}) + \text{cdet}_j \mathbf{A}_{.t} (\mathbf{c}), \end{aligned}$$

where  $\mathbf{b} = (b_1, \dots, b_n)^T \in \mathbf{H}^{n \times 1}$ ,  $\mathbf{c} = (c_1, \dots, c_n)^T \in \mathbf{H}^{n \times 1}$  are arbitrary column-vectors.

**Proposition 2.11** If  $\mathbf{A}^*$  is the Hermitian adjoint matrix (the conjugate transpose) of  $\mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n}$ , then  $\text{rdet}_i \mathbf{A}^* = \overline{\text{cdet}_i \mathbf{A}}$  for all  $i = \overline{1, n}$ .

The following lemma enables to expand  $\text{rdet}_i \mathbf{A}$  by cofactors along the  $i$ -th row for all  $i = \overline{1, n}$ . Consequently, the calculation of the row determinant of a  $n \times n$  matrix is reduced to the calculation of the row determinant of a lower dimension matrix.

**Definition 2.12** Let  $\mathbf{A} \in \mathbf{M}(n, \mathbf{H})$  and  $\text{rdet}_i \mathbf{A} = \sum_j a_{ij} R_{ij}$ , for all  $i = \overline{1, n}$ . Then  $R_{ij}$  is called the right  $ij$ -th cofactor of  $\mathbf{A}$ .

**Lemma 2.13** Let  $R_{ij}$  be the right  $ij$ -th cofactor of  $\mathbf{A} \in M(n, \mathbf{H})$ , that is  $\text{rdet}_i \mathbf{A} = \sum_{j=1}^n a_{ij} \cdot R_{ij}$  for all  $i = \overline{1, n}$ . Then

$$R_{ij} = \begin{cases} -\text{rdet}_k \mathbf{A}_{\cdot j}^{ii}(\mathbf{a}_i), & i \neq j \\ \text{rdet}_k \mathbf{A}^{ii}, & i = j \end{cases} \quad k = \begin{cases} j, & \text{if } i > j; \\ j-1, & \text{if } i < j; \\ \min \{I_n \setminus i\} & \end{cases} \quad (2.1)$$

where  $\mathbf{A}_{\cdot j}^{ii}(\mathbf{a}_i)$  is obtained from  $\mathbf{A}$  by replacing the  $j$ -th column with the  $i$ -th column, and then by deleting both the  $i$ -th row and column,  $I_n = \{1, \dots, n\}$ .

*Proof.* At first we prove that  $R_{ii} = \text{rdet}_k \mathbf{A}^{ii}$ , where  $k = \min \{I_n \setminus i\}$ .

If  $i = 1$ , then  $\text{rdet}_1 \mathbf{A} = a_{11} \cdot R_{11} + a_{12} \cdot R_{12} + \dots + a_{1n} \cdot R_{1n}$ . Consider some monomial of  $\text{rdet}_1 \mathbf{A}$  such that begin with  $a_{11}$  from the left,

$$\begin{aligned} a_{11} \cdot R_{11} &= \sum_{\tilde{\sigma} \in S_n} (-1)^{n-r} a_{11} a_{2i_{k_2}} \dots a_{i_{k_2}+l_2} 2 \dots a_{i_{k_r} i_{k_r}+1} \dots a_{i_{k_r}+l_r i_{k_r}} = \\ &a_{11} \sum_{\tilde{\sigma}_1 \in S_{n-1}} (-1)^{n-1-(r-1)} a_{2i_{k_2}} \dots a_{i_{k_2}+l_2} 2 \dots a_{i_{k_r} i_{k_r}+1} \dots a_{i_{k_r}+l_r i_{k_r}}, \end{aligned}$$

where

$$\begin{aligned} \tilde{\sigma} &= (1) (2 i_{k_2} \dots i_{k_2+l_2}) \dots (i_{k_r} i_{k_r}+1 \dots i_{k_r+l_r}), \\ \tilde{\sigma}_1 &= (2 i_{k_2} \dots i_{k_2+l_2}) \dots (i_{k_r} i_{k_r}+1 \dots i_{k_r+l_r}). \end{aligned}$$

$S_{n-1}$  is the symmetric group on the set  $I_n \setminus 1$ . The numbers of the disjoint cycles and coefficients of every monomial of  $R_{11}$  decrease by one. Since elements of the second row start these monomials on the left and elements of the first row and column do not belong to their, then

$$R_{11} = \sum_{\tilde{\sigma}_1 \in S_{n-1}} (-1)^{n-1-(r-1)} a_{2i_{k_2}} \dots a_{i_{k_2}+l_2} 2 \dots a_{i_{k_r} i_{k_r}+1} \dots a_{i_{k_r}+l_r i_{k_r}} = \text{rdet}_2 \mathbf{A}^{11}. \quad (2.2)$$

If now  $i \neq 1$ , then

$$\text{rdet}_i \mathbf{A} = a_{i1} \cdot R_{i1} + a_{i2} \cdot R_{i2} + \dots + a_{in} \cdot R_{in} \quad (2.3)$$

Consider some monomial of  $\text{rdet}_i \mathbf{A}$  such that begins with  $a_{ii}$  from the left,

$$\begin{aligned} a_{ii} \cdot R_{ii} &= \sum_{\tilde{\sigma} \in S_n} (-1)^{n-r} a_{ii} a_{1i_{k_2}} \dots a_{i_{k_2}+l_2} 1 \dots a_{i_{k_r} i_{k_r}+1} \dots a_{i_{k_r}+l_r i_{k_r}} = \\ &a_{ii} \cdot \sum_{\tilde{\sigma}_1 \in \widehat{S}_{n-1}} (-1)^{n-1-(r-1)} a_{1i_{k_2}} \dots a_{i_{k_2}+l_2} 1 \dots a_{i_{k_r} i_{k_r}+1} \dots a_{i_{k_r}+l_r i_{k_r}}, \end{aligned}$$



where

$$\begin{aligned}\widehat{\sigma} &= (i) (1 i_{k_2} \dots i_{k_2+l_2}) \dots (i_{k_r} i_{k_r+1} \dots i_{k_r+l_r}), \\ \widehat{\sigma}_1 &= (1 i_{k_2} \dots i_{k_2+l_2}) \dots (i_{k_r} i_{k_r+1} \dots i_{k_r+l_r}).\end{aligned}$$

$\widehat{S}_{n-1}$  is the symmetric group on  $I_n \setminus i$ . The numbers of disjoint cycles and the coefficients of every monomial of  $R_{ii}$  again decrease by one. Each monomial of  $R_{ii}$  begins on the left with an entry of the first row. Since elements of the first row start these monomials on the left and elements of the  $i$ -th row and column do not belong to their, then

$$R_{ii} = \sum_{\widehat{\sigma}_1 \in \widehat{S}_{n-1}} (-1)^{n-1-(r-1)} a_{1 i_{k_2}} \dots a_{i_{k_2+l_2} 1} \dots a_{i_{k_r+l_r} i_{k_r}} = \text{rdet}_1 \mathbf{A}^{ii}. \quad (2.4)$$

By combining (2.2) and (2.4), we get  $R_{ii} = \text{rdet}_k \mathbf{A}^{ii}$ ,  $k = \min \{I_n \setminus i\}$ .

Now suppose that  $i \neq j$ . Consider some monomial of  $\text{rdet}_i \mathbf{A}$  in (2.3) such that begins with  $a_{ij}$  from the left,

$$\begin{aligned}a_{ij} \cdot R_{ij} &= \sum_{\bar{\sigma} \in S_n} (-1)^{n-r} a_{ij} a_{j i_{k_1}} \dots a_{i_{k_1+l_1} i} \dots a_{i_{k_r} i_{k_r+1}} \dots a_{i_{k_r+l_r} i_{k_r}} = \\ &= -a_{ij} \cdot \sum_{\bar{\sigma} \in S_n} (-1)^{n-r-1} a_{j i_{k_1}} \dots a_{i_{k_1+l_1} i} \dots a_{i_{k_r} i_{k_r+1}} \dots a_{i_{k_r+l_r} i_{k_r}},\end{aligned}$$

where  $\bar{\sigma} = (i j i_{k_1} \dots i_{k_1+l_1}) \dots (i_{k_r} i_{k_r+1} \dots i_{k_r+l_r})$ . Denote  $\tilde{a}_{i_{k_1+l_1} j} = a_{i_{k_1+l_1} i}$  for all  $i_{k_1+l_1} \in I_n$ . Then

$$a_{ij} \cdot R_{ij} = -a_{ij} \cdot \sum_{\bar{\sigma}_1 \in \widehat{S}_{n-1}} (-1)^{n-r-1} a_{j i_{k_1}} \dots \tilde{a}_{i_{k_1+l_1} j} \dots a_{i_{k_r+l_r} i_{k_r}},$$

where  $\bar{\sigma}_1 = (j i_{k_1} \dots i_{k_1+l_1}) \dots (i_{k_r} i_{k_r+1} \dots i_{k_r+l_r})$ . The permutation  $\bar{\sigma}_1$  does not contain the index  $i$  in each monomial of  $R_{ij}$ . This permutation satisfies the conditions of Definition 2.2 for  $\text{rdet}_j \mathbf{A}_{\cdot j}^{ii}(\mathbf{a}_i)$ . The matrix  $\mathbf{A}_{\cdot j}^{ii}(\mathbf{a}_i)$  is obtained from  $\mathbf{A}$  by replacing the  $j$ -th column with the column  $i$ , and then by deleting both the  $i$ -th row and column. That is,

$$\sum_{\bar{\sigma}_1 \in \widehat{S}_{n-1}} (-1)^{n-r-1} a_{j i_{k_1}} \dots \tilde{a}_{i_{k_1+l_1} j} \dots a_{i_{k_r+l_r} i_{k_r}} = \text{rdet}_j \mathbf{A}_{\cdot j}^{ii}(\mathbf{a}_i)$$

But  $\mathbf{A}_{\cdot j}^{ii}(\mathbf{a}_i)$  is a quadratic matrix of order  $n-1$ . Therefore, more precisely on the set of indices of the matrix  $\mathbf{A}_{\cdot j}^{ii}(\mathbf{a}_i)$  should be noted follows. If  $i > j$ , then the index  $j$  remains the same for  $\mathbf{A}_{\cdot j}^{ii}(\mathbf{a}_i)$  and

$$R_{ij} = -\text{rdet}_j \mathbf{A}_{\cdot j}^{ii}(\mathbf{a}_i) \quad (2.5)$$

But if  $i < j$ , then after deleting both the  $i$ -th row and column in  $\mathbf{A}$  the  $j$ -th row will be the  $j - 1$ -th row of  $\mathbf{A}_{\cdot j}^{ii}(\mathbf{a}_{\cdot i})$ . Therefore,

$$R_{ij} = -\text{rdet}_{j-1} \mathbf{A}_{\cdot j}^{ii}(\mathbf{a}_{\cdot i}) \quad (2.6)$$

Combining (2.5) and (2.6), we finally obtain (2.1).  $\square$

**Definition 2.14** Let  $\mathbf{A} \in \mathbf{M}(n, \mathbf{H})$  and  $\text{cdet}_j \mathbf{A} = \sum_i L_{ij} a_{ij}$ , for all  $j = \overline{1, n}$ . Then  $L_{ij}$  is called the left  $ij$ -th cofactor of  $\mathbf{A}$ .

**Lemma 2.15** Let  $L_{ij}$  be the left  $ij$ th cofactor of of a matrix  $\mathbf{A} \in \mathbf{M}(n, \mathbf{H})$ , that is  $\text{cdet}_j \mathbf{A} = \sum_{i=1}^n L_{ij} \cdot a_{ij}$  for all  $j = \overline{1, n}$ . Then

$$L_{ij} = \begin{cases} -\text{cdet}_k \mathbf{A}_{i \cdot}^{jj}(\mathbf{a}_{j \cdot}), & i \neq j \quad k = \begin{cases} i, & \text{if } j > i; \\ i-1, & \text{if } j < i; \end{cases} \\ \text{cdet}_k \mathbf{A}^{ii}, & i = j \quad k = \min \{J_n \setminus j\} \end{cases} \quad (2.7)$$

where  $\mathbf{A}_{i \cdot}^{jj}(\mathbf{a}_{j \cdot})$  is obtained from  $\mathbf{A}$  by replacing the  $i$ th row with the  $j$ th row, and then by deleting both the  $j$ th row and column,  $J_n = \{1, \dots, n\}$ .

*Proof.* The proof is similar to the proof of Lemma 2.13.  $\square$

If  $\mathbf{A}^* = \mathbf{A}$ , then  $\mathbf{A} \in \mathbf{H}^{n \times n}$  is called a Hermitian matrix. We finish this section by the following theorem which is crucial for row-column determinants of a Hermitian matrix.

**Theorem 2.16** If  $\mathbf{A} \in \mathbf{H}^{n \times n}$  is a Hermitian matrix, then

$$\text{rdet}_1 \mathbf{A} = \dots = \text{rdet}_n \mathbf{A} = \text{cdet}_1 \mathbf{A} = \dots = \text{cdet}_n \mathbf{A} \in \mathbb{R}.$$

By Theorem 2.16, we have the following definition.

**Definition 2.17** Since all column and row determinants of a Hermitian matrix over  $\mathbf{H}$  are equal, we can define the determinant of a Hermitian matrix  $\mathbf{A} \in \mathbf{H}^{n \times n}$ . By definition, we put for all  $i = \overline{1, n}$ ,

$$\det \mathbf{A} := \text{rdet}_i \mathbf{A} = \text{cdet}_i \mathbf{A}.$$

Evidently, if  $\mathbf{A} \in \mathbf{H}^{2 \times 2}$  is Hermitian and  $a_{ij} \in \mathcal{D}(\mathbf{H})$  for all  $i, j = \overline{1, 2}$ , then  $\det \mathbf{A} = 0$ . It would be expected in the general case, but the following example claims that it is not true.

**Example 1** Consider the Hermitian matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1-k & 1-j \\ 1+k & 0 & 1+j \\ 1+j & 1-j & 0 \end{pmatrix}. \quad (2.8)$$

It can easily be checked that  $a_{ij} \in \mathcal{D}(\mathbf{H})$  for all  $i, j = \overline{1, 3}$ . So,

$$\begin{aligned} \det \mathbf{A} &= \text{rdet}_1 \mathbf{A} = \\ &= 0 - 0(1+j)(1-j) + (1-k)(1+j)(1+j) - (1-k)(1+k)0 + \\ &\quad (1-j)(1-j)(1+k) - (1-j)(1+j)0 = 4. \end{aligned}$$

### 3 Properties of the column and row determinants of a Hermitian matrix

**Theorem 3.1** If the matrix  $\mathbf{A}_j.(\mathbf{a}_i.)$  is obtained from a Hermitian matrix  $\mathbf{A} \in M(n, \mathbf{H})$  by replacing its  $j$ -th row with the  $i$ -th row, then for all  $i, j = \overline{1, n}$  such that  $i \neq j$  we have

$$\text{rdet}_j \mathbf{A}_j.(\mathbf{a}_i.) = 0. \quad (3.1)$$

*Proof.* We assume  $n > 3$  for  $\mathbf{A} \in M(n, \mathbf{H})$ . The case  $n \leq 3$  is easily proved by a simple check. Consider some monomial  $d$  of  $\text{rdet}_j \mathbf{A}_j.(\mathbf{a}_i.)$ . Suppose the index permutation of its coefficients forms a direct product of  $r$  disjoint cycles, and denote  $i = i_s$ . Consider all possibilities of disposition of an entry of the  $i_s$ -th row in the monomial  $d$ .

(i) Suppose an entry of the  $i_s$ -th row is placed in  $d$  such that the index  $i_s$  starts some disjoint cycle, i.e.:

$$d = (-1)^{n-r} a_{ji_1} \dots a_{i_k j} u_1 \dots u_\rho a_{i_s i_{s+1}} \dots a_{i_{s+m} i_s} v_1 \dots v_p \quad (3.2)$$

Here we denote by  $u_\tau$  and  $v_t$  products of coefficients whose indices form some disjoint cycles for all  $\tau = \overline{1, \rho}$  and  $t = \overline{1, p}$  such that  $\rho + p = r - 2$  or there are no such products. For  $d$  there are the following three monomials of  $\text{rdet}_j \mathbf{A}_j.(\mathbf{a}_i.)$ .

$$\begin{aligned} d_1 &= (-1)^{n-r} a_{ji_1} \dots a_{i_k j} u_1 \dots u_\rho a_{i_s i_{s+m}} \dots a_{i_{s+1} i_s} v_1 \dots v_p, \\ d_2 &= (-1)^{n-r+1} a_{ji_{s+1}} \dots a_{i_{s+m} i_s} a_{i_s i_1} \dots a_{i_k j} u_1 \dots u_\rho v_1 \dots v_p, \\ d_3 &= (-1)^{n-r+1} a_{ji_{s+m}} \dots a_{i_{s+1} i_s} a_{i_s i_1} \dots a_{i_k j} u_1 \dots u_\rho v_1 \dots v_p. \end{aligned}$$

Suppose  $a_{ji_1} \dots a_{i_k j} = x$  and  $a_{i_s i_{s+1}} \dots a_{i_{s+m} i_s} = y$ , then  $\bar{y} = a_{i_s i_{s+m}} \dots a_{i_{s+1} i_s}$ . Taking into account  $a_{ji_1} = a_{i_s i_1}$ ,  $a_{ji_{s-1}} = a_{i_s i_{s-1}}$  and  $a_{ji_{s+1}} = a_{i_s i_{s+1}}$ , we consider the sum of these monomials.

$$\begin{aligned} d + d_1 + d_2 + d_3 &= (-1)^{n-r} (xu_1 \dots u_\rho y + xu_1 \dots u_\rho \bar{y} - yxu_1 \dots u_\rho - \\ \bar{y} \cdot xu_1 \dots u_\rho) v_1 \dots v_p &= (-1)^{n-r} (xu_1 \dots u_\rho \mathbf{t}(y) - \mathbf{t}(y) xu_1 \dots u_\rho) v_1 \dots v_p = 0. \end{aligned} \quad (3.3)$$

Thus among the monomials of  $\text{rdet}_j \mathbf{A}_j.(\mathbf{a}_i.)$  we find three monomials for  $d$  such that the sum of these monomials and  $d$  is equal to zero.

If in (3.2)  $m = 0$  or  $m = 1$ , we accordingly get such monomials,

$$\begin{aligned} \tilde{d} &= (-1)^{n-r} a_{ji_1} \dots a_{i_k j} u_1 \dots u_\rho a_{i_s i_s} v_1 \dots v_p, \\ \widehat{d} &= (-1)^{n-r} a_{ji_1} \dots a_{i_k j} u_1 \dots u_\rho a_{i_s i_{s+1}} a_{i_{s+1} i_s} v_1 \dots v_p. \end{aligned}$$

For them, there are the following monomials, respectively,

$$\begin{aligned} \tilde{d}_1 &= (-1)^{n-r+1} a_{ji_s} a_{i_s i_1} \dots a_{i_k j} u_1 \dots u_\rho v_1 \dots v_p, \\ \widehat{d}_1 &= (-1)^{n-r+1} a_{ji_{s+1}} a_{i_{s+1} i_s} a_{i_s i_1} \dots a_{i_k j} u_1 \dots u_\rho v_1 \dots v_p. \end{aligned}$$

Taking into account  $a_{ji_1} = a_{i_s i_1}$ ,  $a_{ji_s} = a_{i_s i_s} \in \mathbb{R}$ ,  $a_{ji_{s+1}} = a_{i_s i_{s+1}}$ , and  $a_{i_s i_{s+1}} a_{i_{s+1} i_s} = \mathbf{n}(a_{i_s i_{s+1}}) \in \mathbb{R}$ , we obtain  $\tilde{d} + \tilde{d}_1 = 0$ ,  $\widehat{d} + \widehat{d}_1 = 0$ . Hence, the sums of corresponding two monomials of  $\text{rdet}_j \mathbf{A}_j.(\mathbf{a}_i.)$  are equal to zero in these both cases.

ii) Now suppose that the index  $i_s$  is placed in another disjoint cycle than  $j$  and does not start this cycle,

$$\widetilde{d} = (-1)^{n-r} a_{ji_1} \dots a_{i_k j} u_1 \dots u_\rho a_{i_q i_{q+1}} \dots a_{i_{s-1} i_s} a_{i_s i_{s+1}} \dots a_{i_{q-1} i_q} v_1 \dots v_p.$$

Here we denote by  $u_\tau$  and  $v_t$  products of coefficients whose indices form some disjoint cycles for all  $\tau = \overline{1, \rho}$  and  $t = \overline{1, p}$  such that  $\rho + p = r - 2$  or there are no such products. Now for  $d$  there are the following three monomials of  $\text{rdet}_j \mathbf{A}_j.(\mathbf{a}_i.)$ ,

$$\begin{aligned} \widetilde{d}_1 &= (-1)^{n-r} a_{ji_1} \dots a_{i_k j} u_1 \dots u_\rho a_{i_q i_{q-1}} \dots a_{i_{s+1} i_s} a_{i_s i_{s-1}} \dots a_{i_{q+1} i_q} v_1 \dots v_p, \\ \widetilde{d}_2 &= (-1)^{n-r+1} a_{ji_{s-1}} \dots a_{i_{q+1} i_q} a_{i_q i_{q-1}} \dots a_{i_{s+1} i_s} a_{i_s i_1} \dots a_{i_k j} u_1 \dots u_\rho v_1 \dots v_p, \\ \widetilde{d}_3 &= (-1)^{n-r+1} a_{ji_{s+1}} \dots a_{i_{q-1} i_q} a_{i_q i_{q+1}} \dots a_{i_{s-1} i_s} a_{i_s i_1} \dots a_{i_k j} u_1 \dots u_\rho v_1 \dots v_p. \end{aligned}$$

Assume that

$$\begin{aligned} a_{i_s i_{s+1}} \dots a_{i_{q-1} i_q} &= \varphi, \quad a_{i_q i_{q+1}} \dots a_{i_{s-1} i_s} = \phi, \quad a_{j i_1} \dots a_{i_k j} = x, \\ a_{i_q i_{q+1}} \dots a_{i_{s-1} i_s} a_{i_s i_{s+1}} \dots a_{i_{q-1} i_q} &= y, \quad a_{i_s i_{s+1}} \dots a_{i_{q-1} i_q} a_{i_q i_{q+1}} \dots a_{i_{s-1} i_s} = y_1. \end{aligned}$$

Then we obtain  $y = \phi\varphi$ ,  $y_1 = \varphi\phi$ ,  $\bar{y} = a_{i_q i_{q-1}} \dots a_{i_{s+1} i_s} a_{i_s i_{s-1}} \dots a_{i_{q+1} i_q}$ , and  $\bar{y}_1 = a_{i_s i_{s-1}} \dots a_{i_{q+1} i_q} a_{i_q i_{q-1}} \dots a_{i_{s+1} i_s}$ . Accounting for  $a_{j i_1} = a_{i_s i_1}$ ,  $a_{j i_{s-1}} = a_{i_s i_{s-1}}$ ,  $a_{j i_{s+1}} = a_{i_s i_{s+1}}$ , we have

$$\begin{aligned} \bar{d} + \bar{d}_1 + \bar{d}_2 + \bar{d}_3 &= \\ &= (-1)^{n-r} (xu_1 \dots u_\rho y + xu_1 \dots u_\rho \bar{y} - y_1 xu_1 \dots u_\rho - \bar{y}_1 xu_1 \dots u_\rho) \times \\ &\quad \times v_1 \dots v_p = (-1)^{n-r} (xu_1 \dots u_\rho \mathbf{t}(y) - \mathbf{t}(y_1) xu_1 \dots u_\rho) v_1 \dots v_p = \\ &= (-1)^{n-r} (\mathbf{t}(\phi \cdot \varphi) - \mathbf{t}(\varphi \cdot \phi)) xu_1 \dots u_\rho v_1 \dots v_p. \quad (3.4) \end{aligned}$$

Since by the rearrangement property of the trace,  $\mathbf{t}(\phi \cdot \varphi) = t(\varphi \cdot \phi)$ , then we obtain  $\bar{d} + \bar{d}_1 + \bar{d}_2 + \bar{d}_3 = 0$ .

(iii) If the indices  $i_s$  and  $j$  are placed in the same cycle, then we have the following monomials:  $d_1$ ,  $\tilde{d}_1$ ,  $\widehat{d}_1$  or  $\widetilde{d}_1$ . As shown above, for each of them there are another one or three monomials of  $\text{rdet}_j \mathbf{A}_j(\mathbf{a}_i)$  such that the sums of these two or four corresponding monomials are equal to zero.

We have considered all possible kinds of disposition of an entry of the  $i$ -th row as a factor of some monomial  $d$  of  $\text{rdet}_j \mathbf{A}_j(\mathbf{a}_i)$ . For  $d$ , in each case there exist one or three corresponding monomials such that accordingly the sum of the two or four monomials is equal to zero. Thus, we have (3.1).

We note that if one of factors of  $d$  is zero, then evidently  $d$ ,  $d_1$ ,  $d_2$ ,  $d_3$  are equal 0. If two adjacent factors of  $d$  are adjoint zero divisors (i.e. their product equals zero), then the sums (3.3) or (3.4) contain these adjacent zero divisors as well. Hence, the sums will be equal zero by the same cause.  $\square$

The following theorem can be proved similarly.

**Theorem 3.2** *If the matrix  $\mathbf{A}_i(\mathbf{a}_j)$  is obtained from a Hermitian matrix  $\mathbf{A} \in \mathbf{M}(n, \mathbf{H})$  by replacing of its  $i$ -th column with the  $j$ -th column, then  $\text{cdet}_i \mathbf{A}_i(\mathbf{a}_j) = 0$  for all  $i, j = \overline{1, n}$  such that  $i \neq j$ .*

**Corollary 3.3** *If a Hermitian matrix  $\mathbf{A} \in \mathbf{M}(n, \mathbf{H})$  consists two same rows (columns), then  $\det \mathbf{A} = 0$ .*

*Proof.* Suppose the  $i$ -th row of  $\mathbf{A}$  coincides with the  $j$ -th row, i.e.  $a_{ik} = a_{jk}$  for all  $k \in I_n$  and  $\{i, j\} \in I_n$  such that  $i \neq j$ . Then  $\overline{a_{ik}} = \overline{a_{jk}}$  for all  $k \in I_n$ . Since  $\mathbf{A}$  is Hermitian, then  $a_{ki} = a_{kj}$  for all  $k \in I_n$ , where  $\{i, j\} \in I_n$  and  $i \neq j$ . It means that  $\mathbf{A}$  has two same corresponding columns as well. The matrix  $\mathbf{A}$  may be represented as  $\mathbf{A}_i(\mathbf{a}_j)$ , where  $\mathbf{A}_i(\mathbf{a}_j)$  is obtained from  $\mathbf{A}$  by replacing the  $i$ -th row with the  $j$ -th row. By Theorem 3.1, we have,  $\det \mathbf{A} = \text{rdet}_i \mathbf{A} = \text{rdet}_i \mathbf{A}_i(\mathbf{a}_j) = 0$ .  $\square$

We are needed by the following lemmas.

**Lemma 3.4** [10] *Let  $T_n$  be the sum of all possible products of the  $n$  factors, each of which are either  $h_i \in \mathbf{H}$  or  $\overline{h_i}$  for all  $i = \overline{1, n}$ , by specifying the ordering in the terms,  $T_n = h_1 \cdot h_2 \cdot \dots \cdot h_n + \overline{h_1} \cdot h_2 \cdot \dots \cdot h_n + \dots + \overline{h_1} \cdot \overline{h_2} \cdot \dots \cdot \overline{h_n}$ . Then  $T_n$  consists of the  $2^n$  terms and  $T_n = \mathbf{t}(h_1) \mathbf{t}(h_2) \dots \mathbf{t}(h_n)$ .*

**Lemma 3.5** *If the matrix  $\mathbf{A}_i(\mathbf{a}_i \cdot b)$  is obtained from a Hermitian matrix  $\mathbf{A} \in \mathbf{M}(n, \mathbf{H})$  by right-multiplying of its  $i$ -th column by  $b \in \mathbf{H}$ , then for all  $i = \overline{1, n}$  we have  $\text{rdet}_i \mathbf{A}_i(\mathbf{a}_i \cdot b) = \det \mathbf{A} \cdot b$ .*

*Proof.* Consider some monomial  $d$  of  $\mathbf{A}_i(\mathbf{a}_i \cdot b)$  for  $i = \overline{1, n}$ . Denote  $i_{k_1} := i$ . Then,

$$d = (-1)^{n-r} a_{i_{k_1} i_{k_1+1}} \dots a_{i_{k_1+l_1} i_{k_1}} b a_{i_{k_2} i_{k_2+1}} \dots a_{i_{k_2+l_2} i_{k_2}} \dots \times \\ \times a_{i_{k_r} i_{k_r+1}} \dots a_{i_{k_r+l_r} i_{k_r}} = (-1)^{n-r} h_1 \cdot b \cdot h_2 \cdot \dots \cdot h_r,$$

where  $h_s = a_{i_{k_s} i_{k_s+1}} \dots a_{i_{k_s+l_s} i_{k_s}}$  for all  $s = \overline{1, r}$ . If  $l_s = 1$ , then  $h_s = a_{i_{k_s} i_{k_s+1}} \cdot a_{i_{k_s+1} i_{k_s}} = \mathbf{n}(a_{i_{k_s} i_{k_s+1}}) \in \mathbb{R}$ , and if  $l_s = 0$ , then  $h_s = a_{i_{k_s} i_{k_s}} \in \mathbb{R}$ . Suppose there exists such  $s$  that  $l_s \geq 2$ . By Definition 2.2, the index permutation  $\sigma$  of  $d$  forms a direct products of disjoint cycles and its cycle notation is left-ordered. Denote by  $\sigma_s(i_{k_s}) := (i_{k_s} i_{k_s+1} \dots i_{k_s+l_s})$  a cycle which corresponds to a factor  $h_s$ . Then  $\sigma_s^{-1}(i_{k_s}) := (i_{k_s} i_{k_s+l_s} i_{k_s+1} \dots i_{k_s+1})$  is the cycle which is inverse to  $\sigma_s(i_{k_s})$  and corresponds to the factor  $\overline{h_s}$ . There exist  $2^{p-1}$  monomials of  $\mathbf{A}_i(\mathbf{a}_i \cdot b)$  such that their indices permutations form the direct products of the disjoint cycles  $\sigma_s(i_{k_s})$  or  $\sigma_s^{-1}(i_{k_s})$  for all  $(s = \overline{1, r})$  and keeping their ordering from 1 to  $r$ , where  $p = r - \rho$ , and  $\rho$  is the number of the cycles of the first and second orders. Then by Lemma 3.4 for the sum  $C_1$  of these monomials and  $d$  we obtain,

$$C = (-1)^{n-r} b \cdot \alpha t(h_{\nu_1}) \dots t(h_{\nu_p}),$$

where  $\alpha \in \mathbb{R}$  is a product of factors whose indices form cycles of the first and second orders. Since  $\mathbf{t}(h_{\nu_k}) \in \mathbb{R}$  for all  $\nu_k \in \{1, \dots, r\}$  and  $k = \overline{1, p}$ , then  $b$  commutes with  $\mathbf{t}(h_{\nu_k}) \in \mathbb{R}$  for all  $\nu_k \in \{1, \dots, r\}$  and  $k = \overline{1, p}$ . Therefore,  $\text{rdet}_i \mathbf{A}_{\cdot i}(\mathbf{a}_{\cdot i} \cdot b) = \text{rdet}_i \mathbf{A} \cdot b = b \cdot \det \mathbf{A}$ .  $\square$

**Lemma 3.6** *If  $\mathbf{A}_{i\cdot}(b \cdot \mathbf{a}_{i\cdot})$  is obtained from Hermitian  $\mathbf{A} \in \mathbf{M}(n, \mathbf{H})$  by left-multiplying of its  $i$ -th row by  $b \in \mathbf{H}$ , then for all  $i = \overline{1, n}$  we have*

$$\text{cdet}_i \mathbf{A}_{i\cdot}(b \cdot \mathbf{a}_{i\cdot}) = b \cdot \det \mathbf{A}$$

The proof is similar to the proof of Lemma 3.5.

By Theorems 3.1, 3.2, Lemmas 3.5 and 3.6, and basic properties of the row and column determinants, we have the following theorems.

**Theorem 3.7** *If the  $i$ -th row of a Hermitian matrix  $\mathbf{A} \in \mathbf{M}(n, \mathbf{H})$  is replaced with a left linear combination of its other rows, i.e.  $\mathbf{a}_{i\cdot} = c_1 \mathbf{a}_{i_1\cdot} + \dots + c_k \mathbf{a}_{i_k\cdot}$ , where  $c_l \in \mathbf{H}$  for all  $l = \overline{1, k}$  and  $\{i, i_l\} \subset I_n$ , then*

$$\text{rdet}_i \mathbf{A}_{i\cdot}(c_1 \mathbf{a}_{i_1\cdot} + \dots + c_k \mathbf{a}_{i_k\cdot}) = \text{cdet}_i \mathbf{A}_{i\cdot}(c_1 \mathbf{a}_{i_1\cdot} + \dots + c_k \mathbf{a}_{i_k\cdot}) = 0.$$

**Theorem 3.8** *If the  $j$ -th column of a Hermitian matrix  $\mathbf{A} \in \mathbf{M}(n, \mathbf{H})$  is replaced with a right linear combination of its other columns, i.e.  $\mathbf{a}_{\cdot j} = \mathbf{a}_{\cdot j_1} c_1 + \dots + \mathbf{a}_{\cdot j_k} c_k$ , where  $c_l \in \mathbf{H}$  for all  $l = \overline{1, k}$  and  $\{j, j_l\} \subset J_n$ , then*

$$\text{cdet}_j \mathbf{A}_{\cdot j}(\mathbf{a}_{\cdot j_1} c_1 + \dots + \mathbf{a}_{\cdot j_k} c_k) = \text{rdet}_j \mathbf{A}_{\cdot j}(\mathbf{a}_{\cdot j_1} c_1 + \dots + \mathbf{a}_{\cdot j_k} c_k) = 0.$$

**Definition 3.9** *Let  $\mathbf{a}_{i\cdot} \in \mathbf{H}^{n \times 1}$  for all  $i = \overline{1, m}$ . Row-vectors  $\mathbf{a}_{1\cdot}, \dots, \mathbf{a}_{m\cdot}$  are left linearly dependent, if there exist scalars  $\{b_1, \dots, b_m\} \subset \mathbf{H}$  (which are not all zero) such that  $b_1 \cdot \mathbf{a}_{1\cdot} + \dots + b_m \cdot \mathbf{a}_{m\cdot} = \mathbf{0}$ , where  $\mathbf{0}$  is the zero row vector. If no such scalars exist, then the vectors are said to be left-linearly independent.*

**Definition 3.10** *Let  $\mathbf{a}_{\cdot j} \in \mathbf{H}^{1 \times n}$  for all  $j = \overline{1, m}$ . Column-vectors  $\mathbf{a}_{\cdot 1}, \dots, \mathbf{a}_{\cdot m}$  are right linearly dependent, if there exist scalars  $\{c_1, \dots, c_m\} \subset \mathbf{H}$  (which are not all zero) such that  $\mathbf{a}_{\cdot 1} \cdot c_1 + \dots + \mathbf{a}_{\cdot m} \cdot c_m = \mathbf{0}$ , where  $\mathbf{0}$  is the zero column-vector. If no such scalars exist, then the column-vectors are said to be right-linearly independent.*

By Lemma 3.3, the evident corollary of Theorems 3.7 and 3.8 follows.

**Corollary 3.11** *If the  $i$ -th row of Hermitian  $\mathbf{A} \in M(n, \mathbf{H})$  is a left linear combination of its other rows, or its  $j$ -th column is a right linear combination of its other columns, i.e.  $\exists c_l \in \mathbf{H}$  for  $l = \overline{1, k}$  such that  $\mathbf{a}_{i.} = c_1 \mathbf{a}_{i_1.} + \dots + c_k \mathbf{a}_{i_k.}$  or  $\mathbf{a}_{.i} = \mathbf{a}_{.i_1} c_1 + \dots + \mathbf{a}_{.i_k} c_k$  for  $\{i, i_l\} \subset I_n$ , then  $\det \mathbf{A} = 0$ .*

From Theorems 3.7, 3.8 and basic properties of the row-column determinants for arbitrary matrices, we can obtain the following theorems as well.

**Theorem 3.12** *If the  $i$ -th row of a Hermitian matrix  $\mathbf{A} \in M(n, \mathbf{H})$  is added a left linear combination of its other rows, then*

$$\begin{aligned} \text{rdet}_i \mathbf{A}_{i.} (\mathbf{a}_{i.} + c_1 \cdot \mathbf{a}_{i_1.} + \dots + c_k \cdot \mathbf{a}_{i_k.}) &= \\ = \text{cdet}_i \mathbf{A}_{i.} (\mathbf{a}_{i.} + c_1 \cdot \mathbf{a}_{i_1.} + \dots + c_k \cdot \mathbf{a}_{i_k.}) &= \det \mathbf{A}, \end{aligned}$$

where  $c_l \in \mathbf{H}$  for all  $l = \overline{1, k}$  and  $\{i, i_l\} \subset I_n$ .

**Theorem 3.13** *If the  $j$ -th column of a Hermitian matrix  $\mathbf{A} \in M(n, \mathbf{H})$  is added a right linear combination of its other columns, then*

$$\begin{aligned} \text{cdet}_j \mathbf{A}_{.j} (\mathbf{a}_{.j} + \mathbf{a}_{.j_1} c_1 + \dots + \mathbf{a}_{.j_k} c_k) &= \\ = \text{rdet}_j \mathbf{A}_{.j} (\mathbf{a}_{.j} + \mathbf{a}_{.j_1} c_1 + \dots + \mathbf{a}_{.j_k} c_k) &= \det \mathbf{A}, \end{aligned}$$

where  $c_l \in \mathbf{H}$  for all  $l = \overline{1, k}$  and  $\{j, j_l\} \subset J_n$ .

## 4 Determinantal representations the inverse of a Hermitian matrix

### 4.1 The inverse of a Hermitian matrix

**Theorem 4.1** *If  $\mathbf{A} \in M(n, \mathbf{H})$  is Hermitian and  $\det \mathbf{A} \neq 0$ , then there exist an unique right inverse matrix  $(R\mathbf{A})^{-1}$  and an unique left inverse matrix  $(L\mathbf{A})^{-1}$  of  $\mathbf{A}$ , where  $(R\mathbf{A})^{-1} = (L\mathbf{A})^{-1} =: \mathbf{A}^{-1}$ , and they have the following determinantal representations, respectively,*

$$(R\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} R_{11} & R_{21} & \cdots & R_{n1} \\ R_{12} & R_{22} & \cdots & R_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{pmatrix}, \quad (4.1)$$



$$(L\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} L_{11} & L_{21} & \cdots & L_{n1} \\ L_{12} & L_{22} & \cdots & L_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ L_{1n} & L_{2n} & \cdots & L_{nn} \end{pmatrix}, \quad (4.2)$$

where  $R_{ij}$  and  $L_{ij}$  can be obtained by (2.1) and (2.7), respectively, for all  $i, j = \overline{1, n}$ .

*Proof.* Let  $\mathbf{B} = \mathbf{A} \cdot (R\mathbf{A})^{-1}$ . We obtain the entries of  $\mathbf{B}$  by multiplying matrices. For all  $i = \overline{1, n}$ , we have

$$b_{ii} = (\det \mathbf{A})^{-1} \sum_{j=1}^n a_{ij} \cdot R_{ij} = (\det \mathbf{A})^{-1} \text{rdet}_i \mathbf{A} = \frac{\det \mathbf{A}}{\det \mathbf{A}} = 1,$$

and for all  $i \neq j$

$$b_{ij} = (\det \mathbf{A})^{-1} \sum_{s=1}^n a_{is} \cdot R_{js} = (\det \mathbf{A})^{-1} \text{rdet}_j \mathbf{A}_{j.}(\mathbf{a}_{i.}).$$

If  $i \neq j$ , then by Theorem 3.1  $\text{rdet}_j \mathbf{A}_{j.}(\mathbf{a}_{i.}) = 0$ . Consequently  $b_{ij} = 0$ . Thus  $\mathbf{B} = \mathbf{I}$  and  $(R\mathbf{A})^{-1}$  is the right inverse of the Hermitian matrix  $\mathbf{A}$ .

Suppose  $\mathbf{D} = (L\mathbf{A})^{-1} \mathbf{A}$ . We again get the entries of  $\mathbf{D}$  by multiplying matrices. For all  $i = \overline{1, n}$ ,

$$d_{ii} = (\det \mathbf{A})^{-1} \sum_{j=1}^n L_{ij} \cdot a_{ij} = (\det \mathbf{A})^{-1} \text{cdet}_j \mathbf{A} = \frac{\det \mathbf{A}}{\det \mathbf{A}} = 1,$$

and for all  $i \neq j$ ,

$$d_{ij} = (\det \mathbf{A})^{-1} \sum_{s=1}^n L_{si} \cdot a_{sj} = (\det \mathbf{A})^{-1} \text{cdet}_i \mathbf{A}_{.i}(\mathbf{a}_{.j}).$$

If  $i \neq j$ , then by Theorem 3.2  $\text{cdet}_i \mathbf{A}_{.i}(\mathbf{a}_{.j}) = 0$ . Therefore  $d_{ij} = 0$  for all  $i \neq j$ . Thus  $\mathbf{D} = \mathbf{I}$  and  $(L\mathbf{A})^{-1}$  is the left inverse of the Hermitian matrix  $\mathbf{A}$ .

The equality  $(R\mathbf{A})^{-1} = (L\mathbf{A})^{-1}$  because of the uniqueness of inverses over associative rings.  $\square$

Moreover, the following criterion of invertibility of a Hermitian matrix can be obtained.

**Theorem 4.2** *If  $\mathbf{A} \in M(n, \mathbf{H})$  is Hermitian, then the following propositions are equivalent.*

- i)  $\mathbf{A}$  is invertibility, i.e.  $\mathbf{A} \in GL(n, \mathbf{H})$ ;*
- ii) rows of  $\mathbf{A}$  are left-linearly independent;*
- iii) columns of  $\mathbf{A}$  are right-linearly independent;*
- iiii)  $\det \mathbf{A} \neq 0$ .*

*Proof.* *i)  $\Rightarrow$  ii)* Consider a right system of linear equations  $\mathbf{A} \cdot \mathbf{x} = \mathbf{y}$ . The fact  $\mathbf{A}$  is invertible means that the linear transformation  $\mathbf{A} : \mathbf{x} \rightarrow \mathbf{y}$  is a bijection. Suppose that rows of  $\mathbf{A}$  are left-linearly dependent. It means that  $\exists i \in I_n$  and  $\exists c_l \in \mathbf{H}$  for  $l = \overline{1, k}$  such that  $\mathbf{a}_i = c_1 \mathbf{a}_{i_1} + \dots + c_k \mathbf{a}_{i_k}$ . Then, by elementary row operations the  $i$ -th row reduce to zero, and we lose the bijectivity of the linear transformation  $\mathbf{A}$ . It follows that  $\mathbf{A}$  is non-invertible. Hence, the supposition is false, and rows of  $\mathbf{A}$  are left-linearly independent.

The equivalence *ii)  $\Rightarrow$  iii)* can be proved similarly by considering a left system of linear equations  $\mathbf{x}\mathbf{A} = \mathbf{y}$ .

The equivalences *ii)  $\Rightarrow$  iii)* and *iii)  $\Rightarrow$  iii)* follow from Corollary 3.11.

Finally, the equivalence *iiii)  $\Rightarrow$  i)* is given by Theorem 4.1.  $\square$

**Remark 4.3** *By Theorems 4.2, 3.12 and 3.13, the determinant of a coquaternionic Hermitian matrix satisfy Axioms 1,3 of a noncommutative determinant.*

## 4.2 Cramer's rule for systems of linear coquaternionic equations in Hermitian case

**Theorem 4.4** *Let*

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{y} \tag{4.3}$$

*be a right system of linear equations with a matrix of coefficients  $\mathbf{A} \in M(n, \mathbf{H})$ , a column of constants  $\mathbf{y} = (y_1, \dots, y_n)^T \in \mathbf{H}^{n \times 1}$ , and a column of unknowns  $\mathbf{x} = (x_1, \dots, x_n)^T$ . If  $\mathbf{A}$  is Hermitian and  $\det \mathbf{A} \neq 0$ , then the solution of (4.3) is given by components,*

$$x_j = \frac{\text{c} \det_j \mathbf{A}_j(\mathbf{y})}{\det \mathbf{A}}, \quad j = \overline{1, n}. \tag{4.4}$$

*Proof.* Since  $\det \mathbf{A} \neq 0$ , then, by Theorem 4.1, there exists the unique inverse matrix  $\mathbf{A}^{-1}$ . From this the existence and uniqueness of solutions of (4.3) follows immediately.

By considering  $\mathbf{A}^{-1}$  as the left inverse, the solution of (4.3),  $\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{y}$ , can be represented by components as follows,

$$x_j = (\det \mathbf{A})^{-1} \sum_{i=1}^n L_{ij} \cdot y_i, \quad j = \overline{1, n},$$

where  $L_{ij}$  is the left  $ij$ -th cofactor of  $\mathbf{A}$ . From here (4.4) follows immediately.  $\square$

**Theorem 4.5** *Let*

$$\mathbf{x} \cdot \mathbf{A} = \mathbf{y} \tag{4.5}$$

*be a left system of linear equations with a matrix of coefficients  $\mathbf{A} \in M(n, \mathbf{H})$ , a row of constants  $\mathbf{y} = (y_1, \dots, y_n) \in \mathbf{H}^{1 \times n}$ , and a row of unknowns  $\mathbf{x} = (x_1, \dots, x_n)$ . If  $\mathbf{A}$  is Hermitian and  $\det \mathbf{A} \neq 0$ , then the solution of (4.5) is given by components,*

$$x_i = \frac{\text{rdet}_i \mathbf{A}_i(\mathbf{y})}{\det \mathbf{A}}, \quad i = \overline{1, n}.$$

*Proof.* The proof is similar to the proof of Theorem 4.4 by using (4.1) for determinantal representation of  $\mathbf{A}^{-1}$ .  $\square$

**Example 2** *Let consider a right system of linear equations*

$$\mathbf{A} \mathbf{x} = \mathbf{b} \tag{4.6}$$

*with the matrix  $\mathbf{A}$  from (2.8) and  $\mathbf{b} = (i \ j \ k)^T$ . Since  $\mathbf{A}$  is Hermitian and  $\det \mathbf{A} = 4$ , we can find the solution of (4.6) by Cramer's rule (4.4).*

$$\begin{aligned} x_1 &= \frac{1}{\det \mathbf{A}} \text{cdet}_1 \begin{pmatrix} i & 1-k & 1-j \\ j & 0 & 1+j \\ k & 1-j & 0 \end{pmatrix} = \frac{-3-i+3j+k}{4}, \\ x_2 &= \frac{1}{\det \mathbf{A}} \text{cdet}_2 \begin{pmatrix} 0 & i & 1-j \\ 1+k & j & 1+j \\ 1+j & k & 0 \end{pmatrix} = \frac{1+3i+j-k}{4}, \\ x_3 &= \frac{1}{\det \mathbf{A}} \text{cdet}_3 \begin{pmatrix} 0 & 1-k & i \\ 1+k & 0 & j \\ 1+j & 1-j & k \end{pmatrix} = \frac{2j+2k}{4}. \end{aligned}$$

Now, we shall find the inverse  $\mathbf{A}^{-1}$  of  $\mathbf{A}$  by (4.2).

$$\begin{aligned}
L_{11} &= \text{cdet}_1 \begin{pmatrix} 0 & 1+j \\ 1-j & 0 \end{pmatrix} = 0, L_{12} = -\text{cdet}_1 \begin{pmatrix} 1+k & 1+j \\ 1+j & 0 \end{pmatrix} = 2+2j, \\
L_{13} &= -\text{cdet}_1 \begin{pmatrix} 1+j & 1-j \\ 1+k & 0 \end{pmatrix} = 1+i-j+k, \\
L_{21} &= -\text{cdet}_1 \begin{pmatrix} 1-k & 1-j \\ 1-j & 0 \end{pmatrix} = 2-2j, L_{22} = \text{cdet}_1 \begin{pmatrix} 0 & 1-j \\ 1+j & 0 \end{pmatrix} = 0, \\
L_{23} &= -\text{cdet}_2 \begin{pmatrix} 0 & 1-k \\ 1+j & 1-j \end{pmatrix} = 1+i+j-k, \\
L_{31} &= -\text{cdet}_2 \begin{pmatrix} 0 & 1+j \\ 1-k & 1-j \end{pmatrix} = 1-i+j-k, \\
L_{32} &= -\text{cdet}_2 \begin{pmatrix} 0 & 1-j \\ 1+k & 1+j \end{pmatrix} = 1-i-j+k, \\
L_{33} &= \text{cdet}_2 \begin{pmatrix} 0 & 1-k \\ 1+k & 0 \end{pmatrix} = 0.
\end{aligned}$$

Therefore,

$$\mathbf{A}^{-1} = \frac{1}{4} \begin{pmatrix} 0 & 2-2j & 1-i+j-k \\ 2+2j & 0 & 1-i-j+k \\ 1+i-j+k & 1+i+j-k & 0 \end{pmatrix}.$$

Finally, we see that by the matrix method the identical result is obtained,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 0 & 2-2j & 1-i+j-k \\ 2+2j & 0 & 1-i-j+k \\ 1+i-j+k & 1+i+j-k & 0 \end{pmatrix} \begin{pmatrix} i \\ j \\ k \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -3-i+3j+k \\ 1+3i+j-k \\ 2j+2k \end{pmatrix}.$$

## 5 Cramer's rules for some coquaternionic matrix equations

**Theorem 5.1** *Suppose*

$$\mathbf{A}\mathbf{X}\mathbf{B} = \mathbf{C} \tag{5.1}$$

is a two-sided matrix equation, where  $\mathbf{A} \in \mathbf{H}^{m \times m}$ ,  $\mathbf{B} \in \mathbf{H}^{n \times n}$ ,  $\mathbf{C} \in \mathbf{H}^{m \times n}$  are given,  $\mathbf{X} \in \mathbf{H}^{m \times n}$  is unknown, and  $\mathbf{A}$ ,  $\mathbf{B}$  are Hermitian. If  $\det \mathbf{A} \neq 0$  and  $\det \mathbf{B} \neq 0$ , then the unique solution of (5.1) can be represented as follows,

$$x_{ij} = \frac{\text{rdet}_j \mathbf{B}_{j.} (\mathbf{c}_{i.}^{\mathbf{A}})}{\det \mathbf{A} \cdot \det \mathbf{B}}, \quad (5.2)$$

or

$$x_{ij} = \frac{\text{cdet}_i \mathbf{A}_{.i} (\mathbf{c}_j^{\mathbf{B}})}{\det \mathbf{A} \cdot \det \mathbf{B}}, \quad (5.3)$$

where  $\mathbf{c}_{i.}^{\mathbf{A}} := (\text{cdet}_i \mathbf{A}_{.i} (\mathbf{c}_{1.}), \dots, \text{cdet}_i \mathbf{A}_{.i} (\mathbf{c}_{n.})) \in \mathbf{H}^{n \times 1}$  is the row-vector and  $\mathbf{c}_j^{\mathbf{B}} := (\text{rdet}_j \mathbf{B}_{j.} (\mathbf{c}_{1.}), \dots, \text{rdet}_j \mathbf{B}_{j.} (\mathbf{c}_{m.}))^T \in \mathbf{H}^{1 \times m}$  is the column-vector and  $\mathbf{c}_{i.}$ ,  $\mathbf{c}_{.j}$  are the  $i$ -th row and the  $j$ -th column of  $\mathbf{C}$ , respectively, for all  $i = \overline{1, m}$ ,  $j = \overline{1, n}$ .

*Proof.* By Theorem 4.1,  $\mathbf{A}$  and  $\mathbf{B}$  are invertible. There exists the unique solution of (5.1),  $\mathbf{X} = \mathbf{A}^{-1} \mathbf{C} \mathbf{B}^{-1}$ . If we represent  $\mathbf{A}^{-1}$  as a left inverse by (4.2) and  $(\mathbf{B})^{-1}$  as a right inverse by (4.1), then we have

$$\begin{aligned} \mathbf{X} &= \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} L_{11}^{\mathbf{A}} & L_{21}^{\mathbf{A}} & \dots & L_{m1}^{\mathbf{A}} \\ L_{12}^{\mathbf{A}} & L_{22}^{\mathbf{A}} & \dots & L_{m2}^{\mathbf{A}} \\ \dots & \dots & \dots & \dots \\ L_{1m}^{\mathbf{A}} & L_{2m}^{\mathbf{A}} & \dots & L_{mm}^{\mathbf{A}} \end{pmatrix} \times \\ &\times \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{pmatrix} \frac{1}{\det \mathbf{B}} \begin{pmatrix} R_{11}^{\mathbf{B}} & R_{21}^{\mathbf{B}} & \dots & R_{n1}^{\mathbf{B}} \\ R_{12}^{\mathbf{B}} & R_{22}^{\mathbf{B}} & \dots & R_{n2}^{\mathbf{B}} \\ \dots & \dots & \dots & \dots \\ R_{1n}^{\mathbf{B}} & R_{2n}^{\mathbf{B}} & \dots & R_{nn}^{\mathbf{B}} \end{pmatrix}, \end{aligned}$$

where  $L_{ij}^{\mathbf{A}}$  is a left  $ij$ -th cofactor of  $\mathbf{A}$  for all  $i, j = \overline{1, m}$ , and  $R_{ij}^{\mathbf{B}}$  is a right  $ij$ -th cofactor of  $\mathbf{B}$  for all  $i, j = \overline{1, n}$ . It implies

$$x_{ij} = \frac{\sum_{l=1}^n \left( \sum_{k=1}^m L_{ki}^{\mathbf{A}} c_{kl} \right) R_{jl}^{\mathbf{B}}}{\det \mathbf{A} \cdot \det \mathbf{B}}, \quad (5.4)$$

for all  $i = \overline{1, m}$ ,  $j = \overline{1, n}$ . From this by Definition 2.3, we obtain

$$\sum_{k=1}^m L_{ki}^{\mathbf{A}} c_{kl} = \text{cdet}_i \mathbf{A}_{.i} (\mathbf{c}_{l.}),$$

where  $\mathbf{c}_{.l}$  is the  $l$ -th column of  $\mathbf{C}$  for all  $l = \overline{1, n}$ . Consider the row-vector

$$\mathbf{c}_{i.}^{\mathbf{A}} := (\text{cdet}_{i.} \mathbf{A}_{.i}(\mathbf{c}_{.1}), \dots, \text{cdet}_{i.} \mathbf{A}_{.i}(\mathbf{c}_{.n}))$$

for all  $i = \overline{1, m}$ . By Definition 2.2,  $\sum_{l=1}^n c_{il}^{\mathbf{A}} R_{jl}^{\mathbf{B}} = \text{rdet}_j \mathbf{B}_{j.}(\mathbf{c}_{i.}^{\mathbf{A}})$ , then we get (5.2). Having changed the order of summation in (5.4), we have

$$x_{ij} = \frac{\sum_{k=1}^m L_{ki}^{\mathbf{A}} \left( \sum_{l=1}^n c_{kl} R_{jl}^{\mathbf{B}} \right)}{\det \mathbf{A} \cdot \det \mathbf{B}}.$$

By Definition 2.2, we have  $\sum_{l=1}^n c_{kl} R_{jl}^{\mathbf{B}} = \text{rdet}_j \mathbf{B}_{j.}(\mathbf{c}_{k.})$ , where  $\mathbf{c}_{k.}$  is the  $k$ -th row-vector of  $\mathbf{C}$  for all  $k = \overline{1, n}$ . Denote the following column-vector by

$$\mathbf{c}_{.j}^{\mathbf{B}} := (\text{rdet}_j \mathbf{B}_{j.}(\mathbf{c}_{1.}), \dots, \text{rdet}_j \mathbf{B}_{j.}(\mathbf{c}_{n.}))^T$$

for all  $j = \overline{1, n}$ . By Definition 2.3,  $\sum_{k=1}^n L_{ki}^{\mathbf{A}} c_{kj}^{\mathbf{B}} = \text{cdet}_{i.} \mathbf{A}_{.i}(\mathbf{c}_{.j}^{\mathbf{B}})$ , then we finally have (5.3).  $\square$

If, in (5.1), we put  $\mathbf{A} = \mathbf{I}_m$  or  $\mathbf{B} = \mathbf{I}_n$ , then, respectively, we evidently get the following corollaries.

**Corollary 5.2** *Suppose*

$$\mathbf{A}\mathbf{X} = \mathbf{C} \tag{5.5}$$

*is a right matrix equation, where  $\mathbf{A} \in \mathbf{H}^{m \times m}$ ,  $\mathbf{C} \in \mathbf{H}^{m \times n}$  are given,  $\mathbf{X} \in \mathbf{H}^{m \times n}$  is unknown, and  $\mathbf{A}$  is Hermitian. If  $\det \mathbf{A} \neq 0$ , then the unique solution of (5.5) can be represented as follows,*

$$x_{ij} = \frac{\text{cdet}_{i.} \mathbf{A}_{.i}(\mathbf{c}_{.j})}{\det \mathbf{A}}$$

*where  $\mathbf{c}_{.j}$  is the  $j$ -th column of  $\mathbf{C}$ , for all  $i = \overline{1, m}$ ,  $j = \overline{1, n}$ .*

**Corollary 5.3** *Suppose*

$$\mathbf{X}\mathbf{B} = \mathbf{C} \tag{5.6}$$

*is a left matrix equation, where  $\mathbf{B} \in \mathbf{H}^{n \times n}$ ,  $\mathbf{C} \in \mathbf{H}^{m \times n}$  are given,  $\mathbf{X} \in \mathbf{H}^{m \times n}$  is unknown, and  $\mathbf{B}$  is Hermitian. If  $\det \mathbf{B} \neq 0$ , then the unique solution of (5.6) can be represented as follows,*

$$x_{ij} = \frac{\text{rdet}_j \mathbf{B}_{j.}(\mathbf{c}_{i.})}{\det \mathbf{B}}$$

*where  $\mathbf{c}_{i.}$  is the  $i$ -th row of  $\mathbf{C}$ , for all  $i = \overline{1, m}$ ,  $j = \overline{1, n}$ .*

**Example 3** Let consider the matrix equations

$$\mathbf{A}\mathbf{X}\mathbf{B} = \mathbf{C} \quad (5.7)$$

with the matrix  $\mathbf{A}$  from (2.8),  $\mathbf{B} = \begin{pmatrix} 1 & k \\ -k & 1 \end{pmatrix}$ , and  $\mathbf{C} = \begin{pmatrix} i & 1 \\ 0 & j \\ k & -i \end{pmatrix}$ . Since  $\mathbf{A}$ ,  $\mathbf{B}$  are Hermitian and  $\det \mathbf{A} = 4$ , and  $\det \mathbf{B} = 2$ , we can find the solution of (5.7) by Cramer's rule (5.3). Firstly, we obtain the column-vectors  $\mathbf{c}_j^{\mathbf{B}}$  for  $j = \overline{1, 2}$ . Since

$$c_{11}^{\mathbf{B}} = \text{rdet}_1 \mathbf{B}_{.1}(\mathbf{c}_{.1}) = \text{rdet}_1 \begin{pmatrix} i & 1 \\ -k & 1 \end{pmatrix} = i + k,$$

$$c_{21}^{\mathbf{B}} = \text{rdet}_1 \mathbf{B}_{.1}(\mathbf{c}_{.2}) = \text{rdet}_1 \begin{pmatrix} 0 & j \\ -k & 1 \end{pmatrix} = -i,$$

$$c_{31}^{\mathbf{B}} = \text{rdet}_1 \mathbf{B}_{.1}(\mathbf{c}_{.3}) = \text{rdet}_1 \begin{pmatrix} k & -i \\ -k & 1 \end{pmatrix} = j + k,$$

then  $\mathbf{c}_{.1}^{\mathbf{B}} = \begin{pmatrix} i + k \\ -i \\ j + k \end{pmatrix}$ . Similarly, we get  $\mathbf{c}_{.2}^{\mathbf{B}} = \begin{pmatrix} 1 - j \\ j \\ -1 - i \end{pmatrix}$ . Then by (5.3), we have

$$x_{11} = \frac{\text{cdet}_1 \mathbf{A}_{.1}(\mathbf{c}_{.1}^{\mathbf{B}})}{\det \mathbf{A} \cdot \det \mathbf{B}} = \frac{1}{8} \text{cdet}_1 \begin{pmatrix} i + k & 1 - k & 1 - j \\ -i & 0 & 1 + j \\ j + k & 1 - j & 0 \end{pmatrix} = \frac{-2i + j - k}{4}.$$

Similarly, we obtain

$$x_{12} = \frac{-2 + j + k}{4}, x_{21} = \frac{i + j}{4}, x_{22} = \frac{-1 - k}{4}, x_{31} = \frac{1 + i + j + 3k}{8},$$

$$x_{32} = \frac{3 - 2i - j + 2k}{8}.$$

Finally,

$$\mathbf{X} = \frac{1}{8} \begin{pmatrix} -4i + 2j - 2k & -4 + 2j + 2k \\ 2i + 2j & -2 - 2k \\ 1 + i + j + 3k & 3 - 2i - j + 2k \end{pmatrix}.$$

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